Conformal Type Problem via Discrete Analysis

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Riemann Surfaces

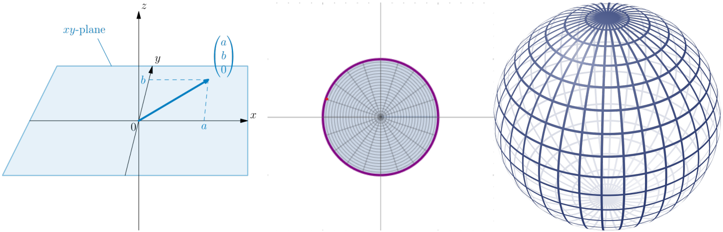

Figure 1: Pictures of Riemann Surfaces

Cylinder amplitudes in 2D Quantum Gravity (Mathijs Wintraecken)

Conformal Type Problem

Uniformization Theorem (Poincaré)

Every simply connected Riemann surface is conformally equivalent to one of the following:

- *the complex plane (Parabolic)*
- *the open unit disk (Hyperbolic)*
- *the Riemann sphere (Elliptic)*

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Conformal Type Problem

The conformal type problem involves classifying Riemann surfaces as parabolic, hyperbolic, or elliptic.

Covering Surfaces

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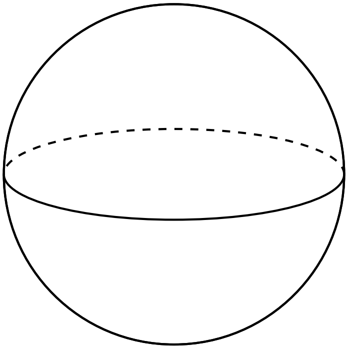

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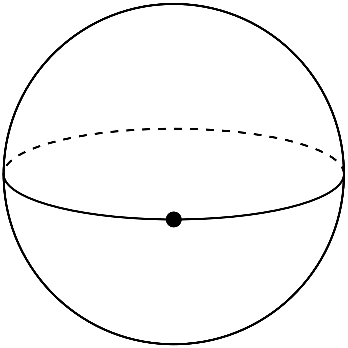

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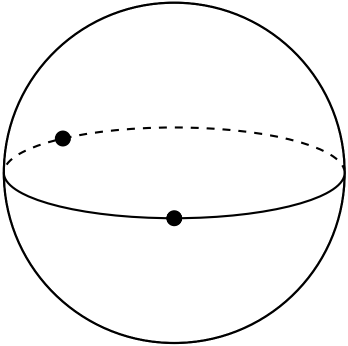

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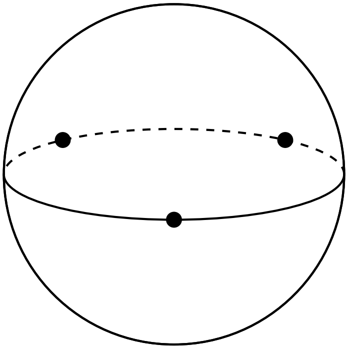

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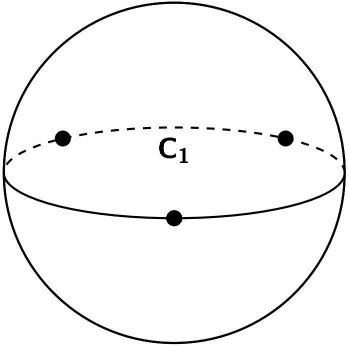

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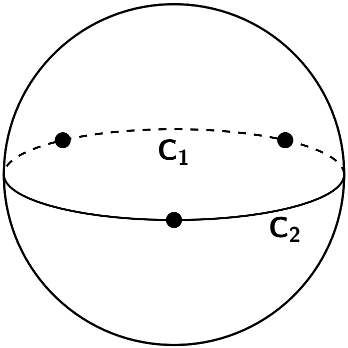

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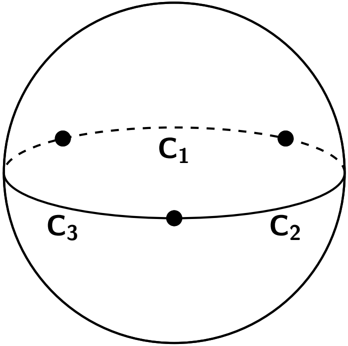

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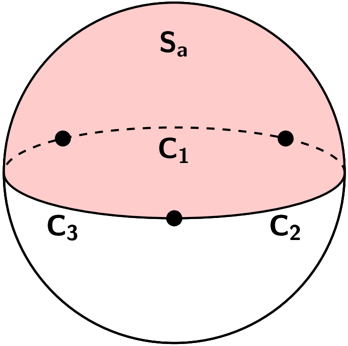

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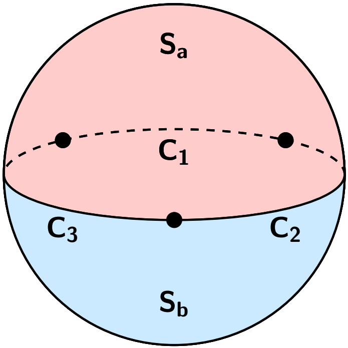
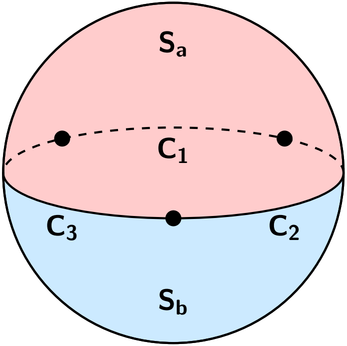


Covering Surface Example

A covering surface consists of sewing copies of red upper-hemispheres and blue lower-hemispheres together.



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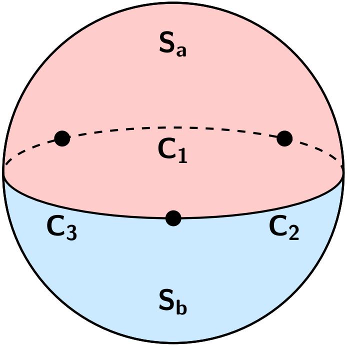

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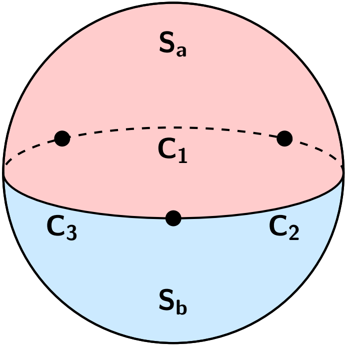

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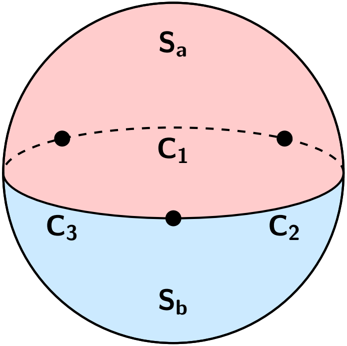

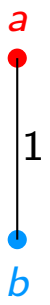

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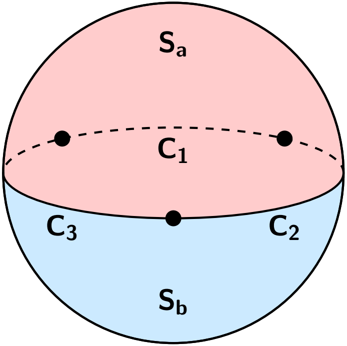

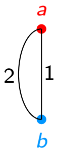

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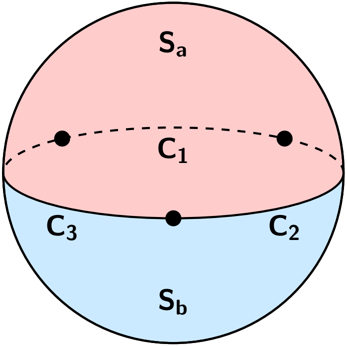
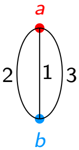
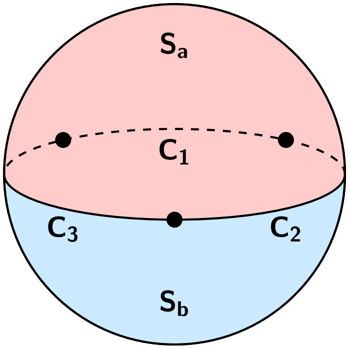




An Infinite Sheeted Covering Surface



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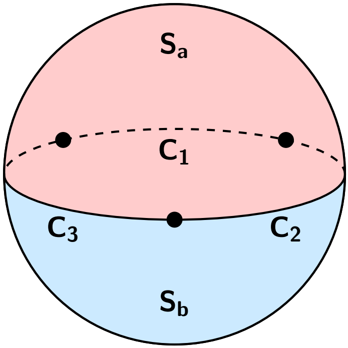

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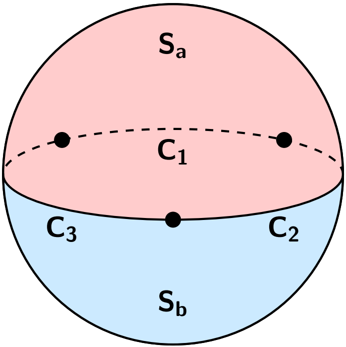

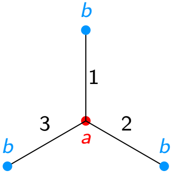

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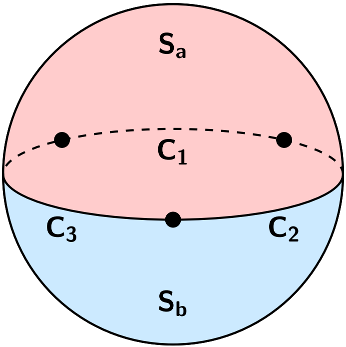

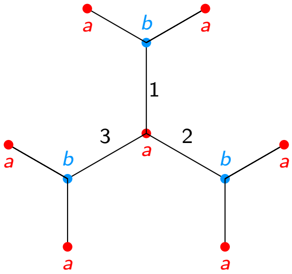

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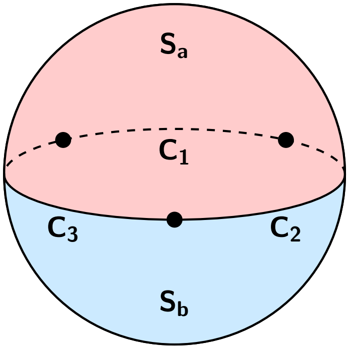

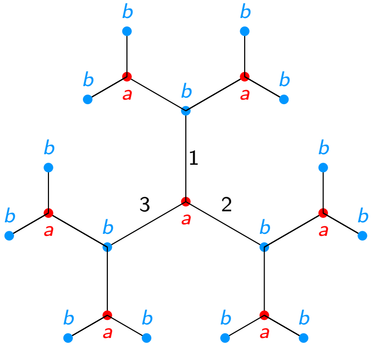

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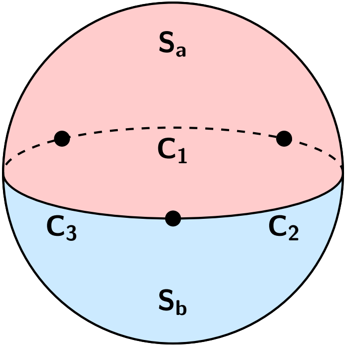
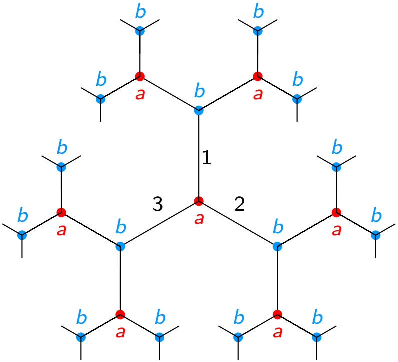




Properties of Speiser Graphs

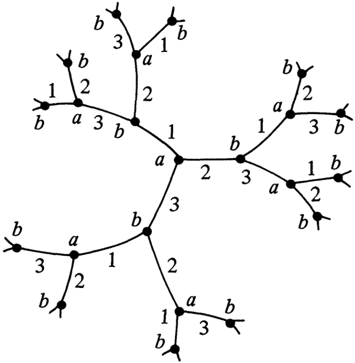
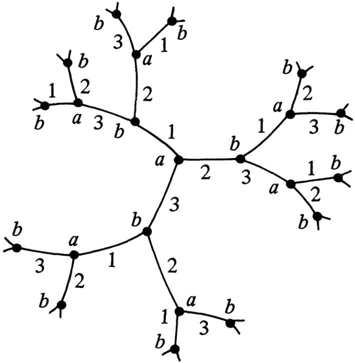


FIGURE 2

RANDOM WALK ON THE SPEISER GRAPH OF A RIEMANN SURFACE (Doyle, 1984)

Properties of Speiser Graphs



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- Vertices alternate between *a* and *b*.

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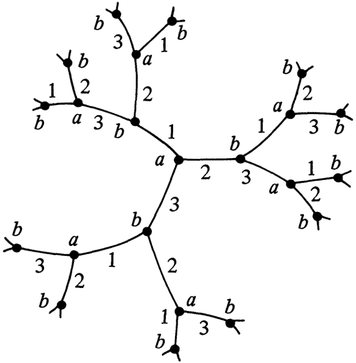

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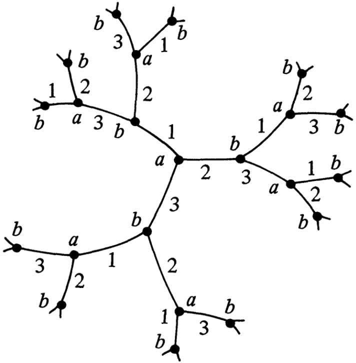

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- Vertices alternate between *a* and *b*.
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- Each edge around the same vertex has a different label

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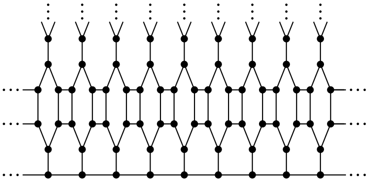
Extended Speiser Graphs



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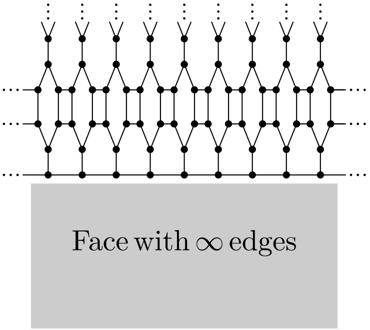

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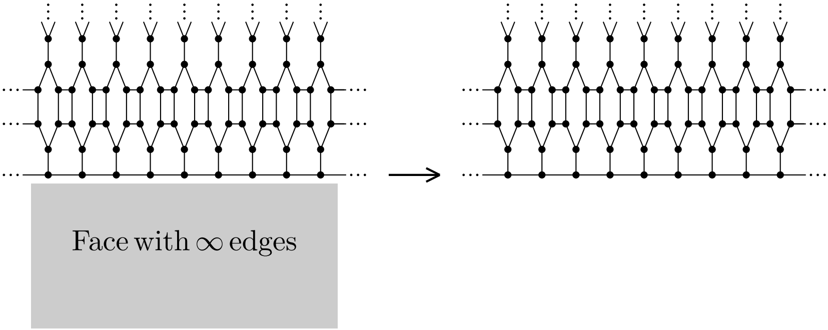

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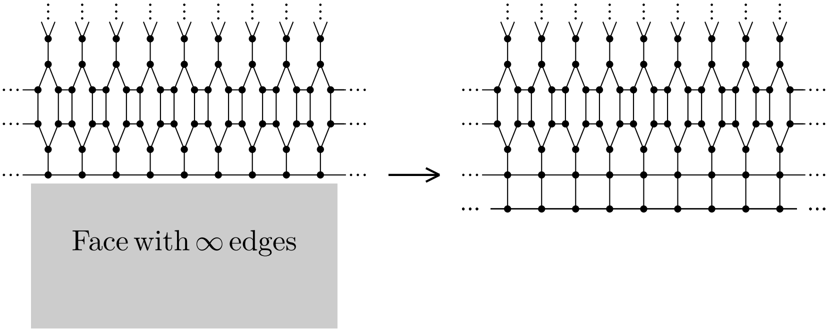

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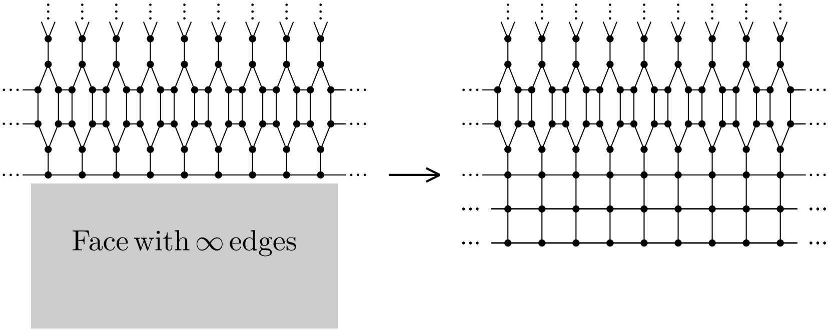

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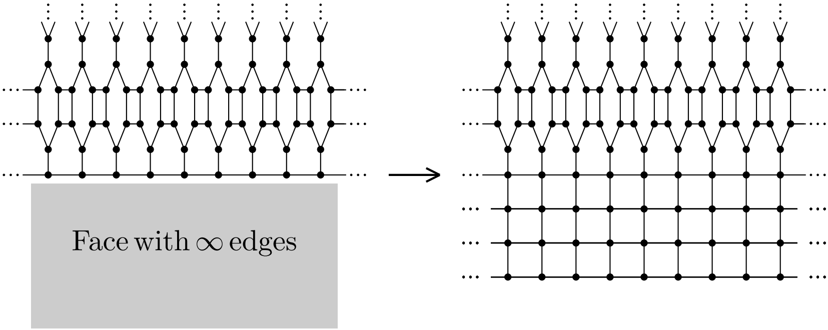

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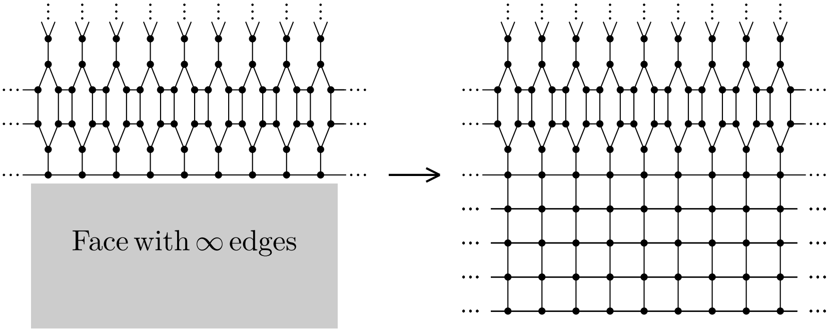

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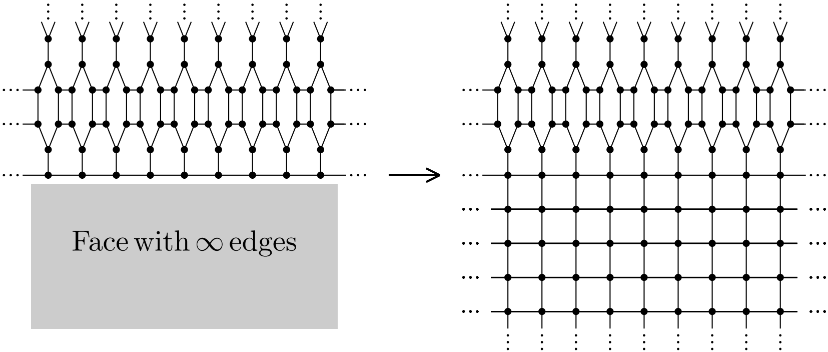

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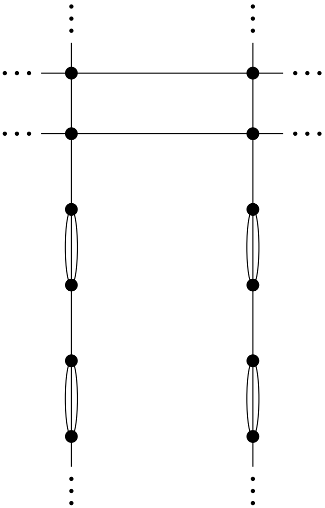

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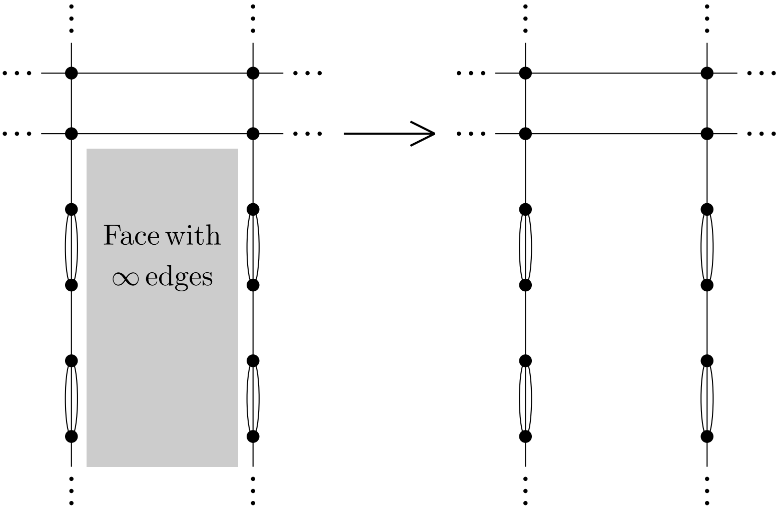

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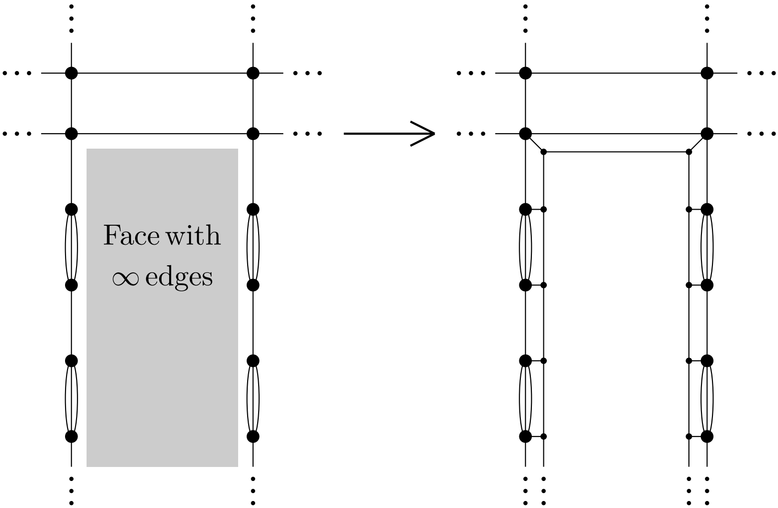

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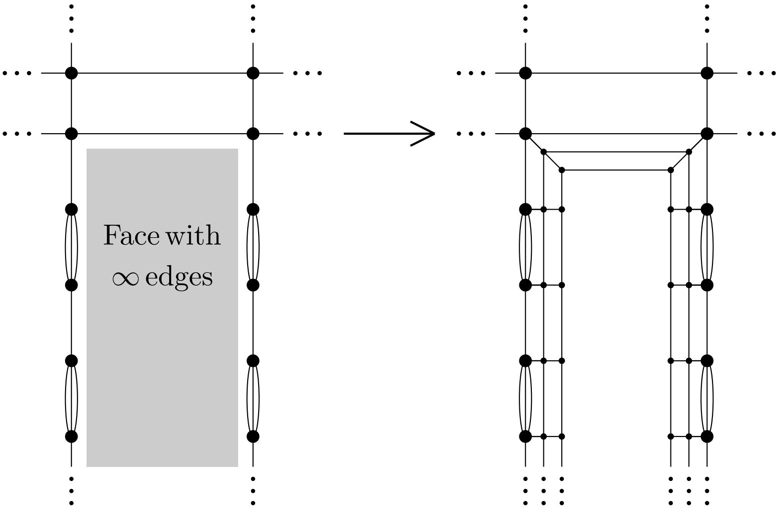

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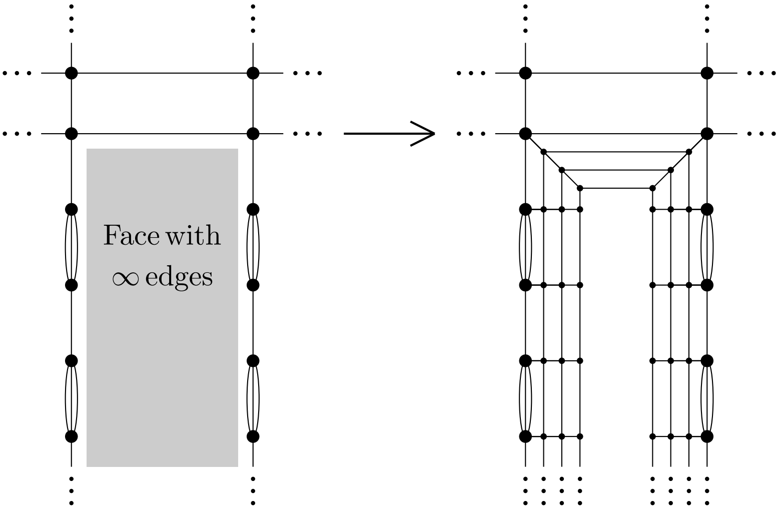

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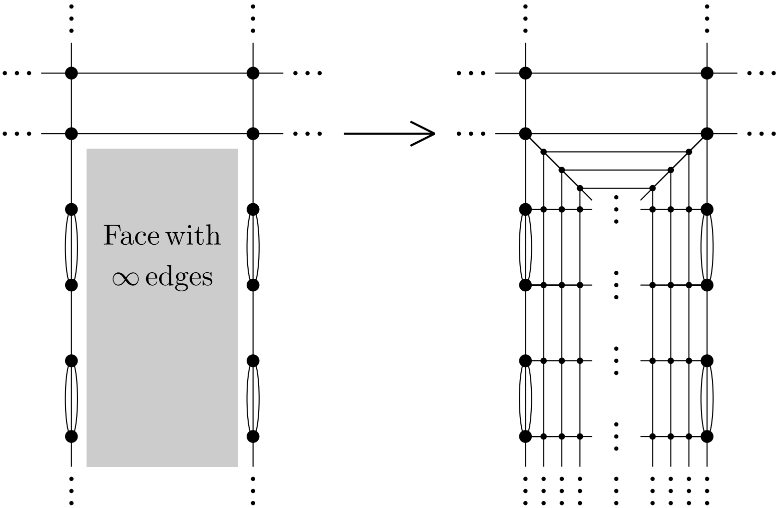

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Two Riemann surfaces can have the same extended Speiser graph!

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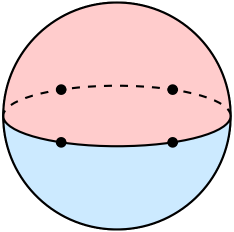

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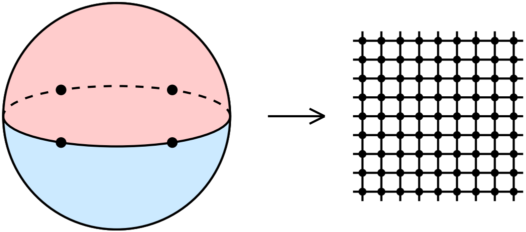

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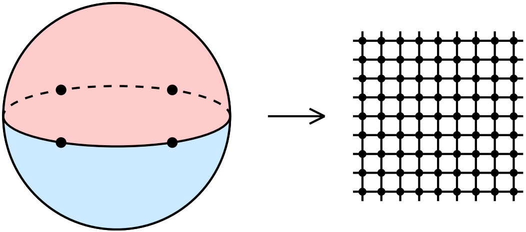

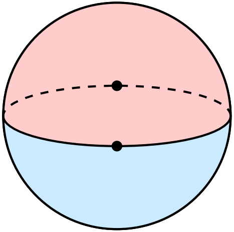

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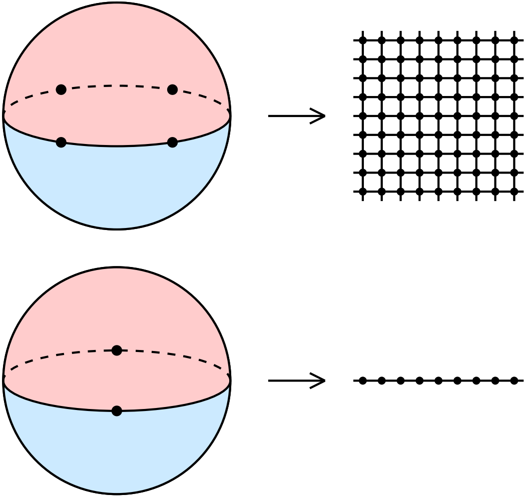

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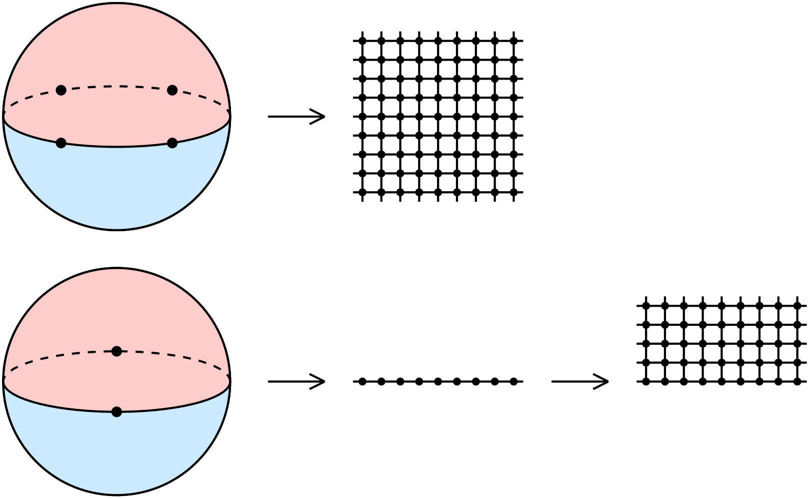

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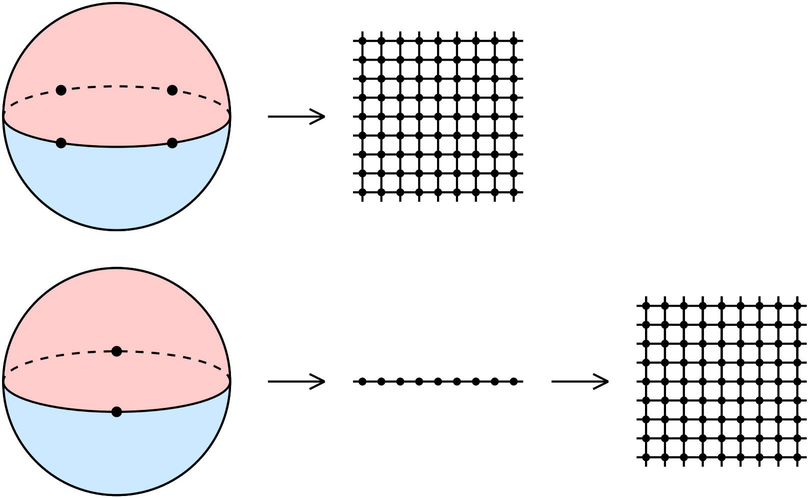

Connecting Surfaces and Graphs

Theorem (Doyle, 1984)

The conformal type (parabolic or hyperbolic) of a covering surface of a Riemann sphere with n punctures is consistent with the type (recurrent or transient respectively) of the random walk on its corresponding extended Speiser graph.

Type Problem for Graphs

Definition

A **random walk** on a graph is defined as follows: starting from a vertex V_0 of the graph called the origin, at each vertex we visit, we randomly travel along one of its edges, each with equal probability. The walk stops if we return to V_0.

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A random walk on a graph is called **recurrent** if it returns to the origin with probability 1.

Definition

A random walk on a graph is called **transient** if it returns to the origin with probability less than 1.

Example of Recurrent Graph

A random walk in 1D is recurrent.

Example of Transient Graph

A random walk on a tree is transient.

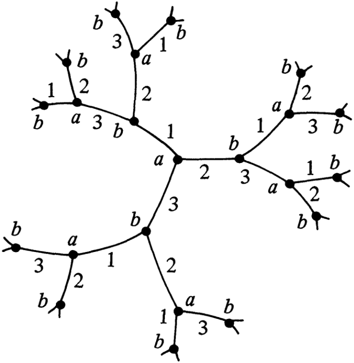

FIGURE 2

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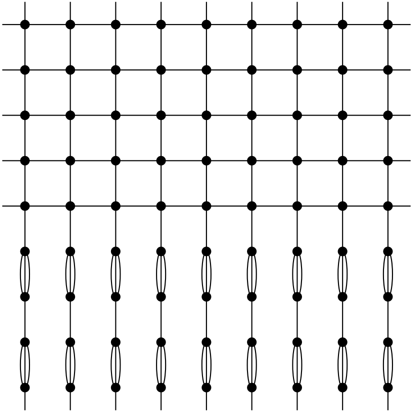
My Research

Theorem (Wang, 2024)

The covering surface of a 4-puncture Riemann sphere corresponding to the Speiser graph depicted on the right has a recurrent extended Speiser graph. Thus, the covering surface is parabolic.



Acknowledgements

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- Dr. Tanya Khovanova, Dr. Slava Gerovitch, Prof. Pavel Etingof
- Friends & family

Bibliography

[1] Peter G. Doyle. Random walk on the Speiser graph of a Riemann surface. *Bulletin (New Series) of the American Mathematical Society*, 11(2):371–377, 1984.

[2] Peter G. Doyle and J. Laurie Snell. Random walks and electric networks, 2000.

[3] Lukas Geyer and Sergei Merenkov. A hyperbolic surface with a square grid net, 2005.

[4] Sergei Merenkov. Determining biholomorphic type of a manifold using combinatorial and algebraic structures, 2013.

[5] P.M. Soardi. *Potential Theory on Infinite Networks*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 1994.

[6] L. I. Volkovyskii. paper investigation of the type problem for a simply connected riemann surface. *Trudy Mat. Inst. Steklov.*, 34:3–171, 1950.